Non-convergence to stability in coalition formation games*

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Abstract

We study the problem of convergence to stability in coalition formation games in which the strategies of each agent are coalitions in which she can participate and outcomes are coalition structures. Given a natural blocking dynamic, an absorbing set is a minimum set of coalition structures that once reached is never abandoned. The coexistence of trivial (singleton) and non-trivial absorbing sets is what causes lack of convergence to stability. To characterize games in which both types of set are present, we first relate circularity among coalitions in preferences (rings) with circularity among coalition structures (cycles) and show that there is a ring in preferences if and only if there is a cycle in coalition structures. Then we identify a special configuration of overlapping rings in preferences characterizing games that lack convergence to stability. Finally, we apply our findings to the study of games induced by sharing rules.

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1 Introduction

The allocation of resources is a core question in economics and the literature on matching has recently emerged as one of the most successful and policy-relevant applications of economic theory: Understanding and management of school choice, kidney exchange and externalities have been enhanced by the insights provided by a wide variety of matching models. From a theoretical perspective, all these problems can be formalized as coalition formation games. In such games, strategies for each agent consist of the set of coalitions in which she may participate and the outcome is a coalition structure, i.e. a partition of the set of agents into coalitions. Coalition formation games encompass a large array of models studied in the literature. Depending on what coalitions are permissible, these games include one-sided problems such as the roommate problem and two-sided problems running from the classical one-to-one marriage problem to many-to-one matching problems with peer effects and complementarities.[1]

In the study of coalition formation games, two different (but closely related) questions arise: a static one that seeks to predict the equilibria of the game; and a dynamic one that analyzes the convergence to those equilibria. In answering the static question of what coalition structures will form, the most appealing equilibrium notion for these games is that of (core) stability. A coalition structure is *stable* if there is no coalition whose members prefer it to the one that they belong to in the coalition structure. A game with (at least) one stable coalition structure is called a stable coalition formation game. Once stability is guaranteed, the dynamic perspective becomes salient. From a market design point of view, this means studying a "natural" process of coalition formation which seeks to mimic the way in which agents would form groups in an environment without a social planner. In cases where decentralized decision making in itself may not suffice to reach a stable outcome, a centralized coordinating pro-

[1] Roth and Sotomayor (1990) is a classic survey of theory, empirical evidence, and design applications of many-to-one matching models.

cess must be imposed in order to attain that outcome. Decentralized processes can be formalized through (myopic) blocking dynamics among coalition structures.[2] In our dynamics, a new coalition structure is formed when a coalition blocks one or more coalitions of a previous coalition structure, and abandoned agents remain single in the new one. A coalition formation game exhibits *convergence to stability* if, starting from any coalition structure, the blocking dynamics lead towards a stable coalition structure. Hence, identifying what games exhibit convergence to stability crucially affects our insights on the implications of different alternatives for market design.

This paper sets out to shed light on the problem of convergence to stability in general coalition formation games. To that end, we study circularity among coalitions in preferences, which we call *rings*, and characterize games which lack convergence to stability in terms of unions of overlapping rings, which we call *ring components.* Crucial to our findings is the concept of *absorbing set*. An absorbing set is a minimal collection of coalition structures that, once entered throughout a blocking dynamics, is never left. In this terminology, a stable coalition structure can be identified with a *trivial* (singleton) absorbing set. Any coalition formation game has at least one, (possibly *non-trivial*) absorbing set.

Marriage problems are particular coalition formation games in which permissible coalitions consist only of singletons and pairs, the set of agents consists of two disjoint subsets, and every agent in each subset prefers staying alone to being matched with another agent in the subset that she belongs to. These problems are always stable games (Gale and Shapley, 1962). Roth and Vande Vate (1990) show that convergence to stability is satisfied for the natural blocking dynamics mentioned above, which means that these games only present trivial absorbing sets. Roommate problems can be seen as generalizations of marriage problems with the same permissible coalitions but without the two-sided restriction on agents. Here, the blocking dynamics can have more complicated patterns. Notably, a roommate problem can have *either* trivial absorbing sets *or*

[2] Another possibility is to consider *farsightedness* in the blocking dynamics, see for example Diamantoudi and Xue (2003) and Ray and Vohra (2015a,b) among others.

non-trivial absorbing sets. Tan (1991) establishes the necessary and sufficient conditions for a problem to be of one type or the other (see also Inarra et al., 2013). For those problems in which absorbing sets are trivial, our blocking dynamics ensure convergence to stability. For those problems in which absorbing sets are non-trivial, the profile of agents' preferences exhibits rings and there is no stable coalition structure. However, for general coalition formation games, it is the coexistence of both trivial and non-trivial absorbing sets that causes lack of convergence to stability (Proposition 1). From this perspective, our contribution consists of a characterization of those games in which both trivial and non-trivial absorbing sets are present. To derive our characterization result, we elaborate on the idea of circularity among coalitions. Our first observation is that the blocking dynamics can generate cycles of coalition structures. We show –Theorem 1– that each cycle of coalition structures induces a ring in preferences and, conversely, every ring in preferences induces a cycle of coalition structures. However, a ring in preferences is not a robust enough notion to create a non-trivial absorbing set. The reasons for this are two-fold: the coalitions that form the ring may collapse into a stable coalition structure and coalition structures formed with ring coalitions (and single agents) can be blocked by coalitions that include agents not in the ring.

Theorem 2 presents our main characterization. If the configuration of coalitions in the profile of preferences and in the blocking dynamics allows agents in a ring component to circulate between its coalitions, and *only* between those coalitions, then a non-trivial absorbing set is obtained. Conversely, in any non-trivial absorbing set it is possible to identify coalitions that form a ring component with such features. Therefore, the existence of a ring component of this type, which we call *effective*, is equivalent to the lack of convergence to stability.

As an application of our results, we analyze some economic environments in which coalitions produce an output to be divided among their members according to a pre-specified sharing rule. In such environments, the sharing rule naturally induces a game where each agent ranks the coalitions to which she belongs according to the payoffs that she gets. Here, the question to be answered is what rules generate stable coalition formation games in which decentralized decision-making leads to a stable coalition structure. We focus on two types of sharing rule: Bargaining rules and rationing rules. We show that games induced by *pairwise aligned* bargaining rules (Pycia, 2012), which include the Nash bargaining rule (Nash, 1950), exhibit convergence to stability (Theorem 3). A similar result is obtained in the context of rationing for *parametric* rules (see Young, 1987; Stovall, 2014), which include several of the most thoroughly-studied rules in the rationing literature (Theorem 4).

Finally, to analyze some real life situations such as academic labor market in which complementarities are relevant, we introduce a sufficient condition for a game to lack convergence to stability. That condition defines a class of games in which agents' preferences feature rings, and the (unique) stable coalition structure of the game is surrounded by one of those rings.

Related literature

In the literature on coalition formation games the papers by Banerjee et al. (2001), Bogomolnaia and Jackson (2002) and Iehlé (2007) identify structures of preferences that guarantee the existence of stable coalition structures.[3] Echenique and Yenmez (2007) develop an algorithm for matching markets with preferences over colleagues to determine the existence of stable matchings. Furthermore, Pycia (2012) and Gallo and Inarra (2018), in different contexts, study what sharing rules induce stable coalition formation games.

The notion of absorbing sets has been studied in different contexts and under different names: By Inarra et al. (2013) for the roommate problem, by Olaizola and Valenciano (2014) and Jackson and Watts (2002) in a network context, (in the latter under the name of "closed cycles"). As far as we know, Schwartz (1970) was the first to introduce this notion for collective decision making prob-

[3]Coalition formation games were first studied by Drezé and Greenberg (1980) under the name of *hedonic games*.

lems and Shenoy (1979, 1980) proposed it under the name of "elementary dynamic solution" for n-person cooperative games. Furthermore, the union of absorbing sets gives the "admissible set" (Kalai and Schmeidler, 1977), a solution defined for abstract systems and applied to various bargaining situations. Recently, Demuynck et al. (2019) define a closely related notion, the "myopic stable set", in a very general class of social environments and study its relation to other solution concepts.

Some papers have studied whether there are decentralized matching markets that converge to stability. The aforementioned procedure of Roth and Vande Vate (1990) for the marriage problem was generalized by Chung (2000) for the roommate problem with weak preferences. Later, Klaus and Klijn (2005) extend it for many-to-one matching with couples and Kojima and Ünver (2008) for many-to-many matching problems.

Eriksson and Häggström (2008) show that a stable matching can be attained by means of a decentralized market, even in cases of incomplete information in two-sided matching.[4] Following a different approach, Diamantoudi et al. (2004) analyze convergence to stability in the stable roommate problem with strict preferences. In that paper, a stable matching is fixed and starting from any matching a path to stability is constructed by trying to get the pairs in the fixed matching until a stable matching (possibly another) is reached. All the above works study the same natural blocking dynamics that we study in this paper, in which abandoned agents are left single when a new coalition is obtained through blocking. A different approach is taken by Tamura (1993) in the marriage problem. Following Knuth (1976), he considers problems with equal numbers of men and women, all of them mutually acceptable, in which all agents are always matched. Unlike the standard blocking dynamics, the less realistic dynamics that he uses assume that when a couple satisfies a blocking pair the abandoned partners also match to each other. Knuth sets the ques-

[4] In a context of overlapping group structures, paths to stability are analyzed by Mauleon et al. (2019).

tion of whether there is convergence to stability in this model[5] and Tamura gives a counter-example in which some matchings cannot converge to any stable matching. The example shows the coexistence of five absorbing sets of cardinality one and one of cardinality sixteen.

However, to the best of our knowledge, there are no published works dealing with convergence to stability in the entire class of coalition formation games.[6] Furthermore, our analysis of convergence to stability differs from that of the papers mentioned. We do not outline a specific procedure to reach stability or fix a stable matching to come after. Instead, we study the stable coalition formation games that have non-trivial absorbing sets and therefore lack convergence to stability.

The rest of the paper is organized as follows. Section 2 presents the model, the notion of absorbing set, and links the lack of convergence to stability with the co-existence of trivial and non-trivial absorbing sets. Section 3 studies the relation between rings in the profile of preferences and cycles of the coalition structures. Section 4 sets out the definition of ring component. This enables us to establish our characterization result. Section 5 applies some previous results to study coalition formation games induced by sharing rules. Section 6 introduces the class of enclosed coalition formation games in which non-convergence to stability is guaranteed. Some concluding remarks are given in Section 7.

2 Coalition formation systems and absorbing sets

In this section, we first introduce the preliminaries of the paper and then present the notions of coalition formation system and absorbing set.

Let $N = \{1, \ldots, n\}$ be a finite set of *agents*. A non-empty subset C of N is called a *coalition*. Let $\mathcal{K}$ denote the set of *permissible* coalitions. Assume that

[5] This is open problem number 8 in Knuth (1976).

[6] There is an unpublished manuscript by Pápai (2003) that addresses this problem.

$\{i\} \in \mathcal{K}$ for each $i \in N$.[7] It is natural to focus only on permissible coalitions, since in most contexts agents cannot be coerced to form all coalitions. Each agent $i \in N$ has a strict, transitive *preference relation* over the set of permissible coalitions of $\mathcal{K}$ to which she belongs, denoted by $\succ_i$, such that $i \in C \cap C'$ and $C \succ_i C'$ mean that agent i prefers coalition C to C'. From now on, when we write $C \succ_i C'$ it is understood that i belongs to $C' \cap C$. Throughout the paper, we assume that for each non-single coalition $C \in \mathcal{K}$ and, for each $i \in C$, $C \succ_i \{i\}$. A preference profile of all agents over permissible coalitions, $\succ_N = (\succ_i)_{i \in N}$, defines a *coalition formation game* which is denoted by $(N, \mathcal{K}, \succ_N)$. Let Π denote the set of partitions of N into permissible coalitions, which we call *coalition structures.* A generic element of Π is denoted by π. For each $\pi \in \Pi$, $\pi(i)$ denotes the coalition in π that contains agent i. Given $C \in \mathcal{K}$ and $\pi \in \Pi$, C is said to *block* π if $C \succ_i \pi(i)$ for all $i \in C$.

The main solution concept for a coalition formation game is that of core stability, namely a coalition structure that is immune to deviation of coalitions. In such games, a coalition structure $\pi \in \Pi$ is *stable* if no coalition blocks it. Hereafter, a stable coalition structure is denoted by π^N. Since we are interested in convergence to stability, throughout this paper we focus only on *stable coalition formation games*, i.e. games with a non-empty core.

2.1 Coalition formation systems

As just mentioned, a stable coalition structure is immune to any coalitional blocking. But if a coalition structure is not stable then its blocking by a coalition does not specify its transformation into a new coalition structure. However, the analysis of convergence to stability requires the definition of some blocking dynamics between coalition structures. To that end, we associate a coalition formation system with a coalition formation game. The associated system is a pair

[7] Restricting to permissible coalitions is commonplace in game theory literature (for instance, see Kalai et al., 1979; Myerson, 1977), and in particular in coalition formation games (see Pápai, 2003; Inal, 2015).

formed by the set of coalition structures that can be formed with the permissible coalitions defined in the game and a binary relation which drives transition from one coalition structure to another. By doing this, we specify the concept of (lack of) convergence to stability.

The binary relation chosen is consistent with the standard blocking definition in that all members of the blocking coalition strictly improve. However, once a coalition structure has been blocked there is no single way to define how the new coalition structure emerges. If one or more agents leave a coalition, what happens with the remaining agents? Do they become singletons or do they remain together? Hart and Kurz (1983) argue that if a coalition is an agreement of all its members and then some agents leave, the agreement breaks down and the remaining agents become singletons. In our analysis this assumption fits well, because our modeling only considers coalitions which are permissible, and the coalition of abandoned agents might not be permissible once a new coalition is formed.

Definition 1 *Let $(N, \mathcal{K}, \succ_N)$ be a coalition formation game. The* ***blocking relation*** *$\gg$ over Π is defined as follows: $\pi' \gg \pi$ if and only if there is $C \in \mathcal{K}$ such that*

(i) $C \in \pi'$ and C blocks π,

(ii) for each $C' \in \pi$ such that $C' \cap C \neq \emptyset$, $\pi'(j) = \{j\}$ for each $j \in C' \setminus C$,

(iii) for each $C' \in \pi$ such that $C' \cap C = \emptyset$, $C' \in \pi'$.

The pair $(\mathbf{\Pi}, \gg)$ *is called the* ***coalition formation system*** *associated with the coalition formation game $(N, \mathcal{K}, \succ_N)$. When we want to stress the role of coalition C, we say that* $\boldsymbol{\pi' \gg \pi}$ ***via C****.*

Condition (i) says that each agent i of the permissible coalition C improves in π' with respect to her position in π. Condition (ii) says that permissible coalitions from which one or more agents depart break into singletons in π'. Condition (iii) says that the permissible coalitions that do not suffer any departure in π, remain unchanged in π'. Notice that the blocking relation $\gg$ implies that agents behave myopically, in the sense that they take the decision about blocking a coalition structure by considering just the resulting coalition, i.e. they are unable to foresee their positions even one step ahead.[8]

Remark 1 *The blocking relation $\gg$ is irreflexive, antisymmetric and not necessarily transitive.*

Given $\gg$, let $\gg^T$ be the *transitive closure* of $\gg$. That is, $\pi' \gg^T \pi$ if and only if there is a finite sequence of coalition structures $\pi = \pi^0, \pi^1, \ldots, \pi^J = \pi'$ such that, for all $j \in \{1, ..., J\}$, $\pi^j \gg \pi^{j-1}$. Hereafter, we say that coalition formation game $(N, \mathcal{K}, \succ_N)$ *exhibits convergence to stability* if for each $\pi \in \Pi$, there is a stable coalition structure $\pi^N \in \Pi$ such that $\pi^N \gg^T \pi$. Otherwise, we say that the game *lacks convergence to stability*.

2.2 Absorbing sets

Our tool for studying lack of convergence to stability is the notion of absorbing set, which is a minimal set of coalition structures that once entered through the blocking relation is never left. An appealing property of absorbing sets is that each coalition formation system has at least one, although in general it may not be unique.

Definition 2 *Let $(N, \mathcal{K}, \succ_N)$ be a coalition formation game. A non-empty set of coalition structures $\mathcal{A}^N \subseteq \Pi$ is an **absorbing set** whenever for each $\pi \in \mathcal{A}^N$ and each $\pi' \in \Pi \setminus \{\pi\}$,*

$$\pi' \gg^T \pi \text{ if and only if } \pi' \in \mathcal{A}^N.$$

*If $|\mathcal{A}^N| \geq 3$, $\mathcal{A}^N$ is said to be a **non-trivial absorbing set**. Otherwise, the absorbing set is **trivial**.*

Notice that coalition structures in $\mathcal{A}^N$ are symmetrically connected by the relation $\gg^T$, and that no coalition structure in $\mathcal{A}^N$ is dominated by a coalition structure that is not in the set. Next, we introduce a remark containing four facts about absorbing sets.

[8] From now on, it is understood that all coalitions considered here are permissible ones.

Remark 2 *Facts on absorbing sets.*

(i) *An absorbing set $\mathcal{A}^N$ contains no stable coalition structure if and only if* $|\mathcal{A}^N| \geq 3$.

(ii) π^N *is a stable coalition structure if and only if* $\{\pi^N\}$ *is an absorbing set.*

(iii) *For each non-stable coalition structure* $\pi \in \Pi$, *there are an absorbing set* $\mathcal{A}^N$ *and a coalition structure* $\pi' \in \mathcal{A}^N$ *such that* $\pi' \gg^T \pi$.

(iv) *For each absorbing set* $\mathcal{A}^N$, *either* $|\mathcal{A}^N| = 1$ *or* $|\mathcal{A}^N| \geq 3$.

Remark 2 (i) is implied by the antisymmetry of $\gg$. Remark 2 (ii) recalls that each stable coalition structure is in itself an absorbing set. Remark 2 (iii) says that from any non-stable coalition structure there is a finite sequence of such structures that reaches a coalition structure of an absorbing set (this property is called outer stability in Kalai and Schmeidler (1977)). Remark 2 (iv) is straightforwardly implied by (i) and (ii). This section concludes with a proposition that relates stability and absorbing sets.

Proposition 1 *A stable coalition formation game lacks convergence to stability if and only if its associated coalition formation system has a non-trivial absorbing set.*

Proof. Let $(N, \mathcal{K}, \succ_N)$ be a coalition formation game and let $(\Pi, \gg)$ be its associated coalition formation system.
($\Longrightarrow$) Assume that $(\Pi, \gg)$ does not have non-trivial absorbing sets. Then, by Remark 2 (ii), the only element of each absorbing set is a stable coalition structure. By Remark 2 (iii), for each non-stable coalition structure $\pi \in \Pi$ there is a stable coalition structure π^N such that $\pi^N \gg^T \pi$. Therefore, coalition formation game $(N, \mathcal{K}, \succ_N)$ exhibits convergence to stability.
($\Longleftarrow$) Assume that $(\Pi, \gg)$ has a non-trivial absorbing set $\mathcal{A}^N$ and let $\pi \in \mathcal{A}^N$. Then, by Remark 2 (i), $\mathcal{A}^N$ has no stable coalition structure. Therefore, by the definition of absorbing set, there is no stable coalition structure π^N such that

$\pi^N \gg^T \pi$. This means that coalition formation game $(N, \mathcal{K}, \succ_N)$ lacks convergence to stability. $\square$

3 Rings and cycles

This section relates the notions of rings and cycles. In this paper, "cycle" refers to the circularity of coalition structures in a coalition formation system. First, some notation and the definition of ring must be introduced. For each pair $C, C' \subseteq N$ such that $C \cap C' \neq \emptyset$, $C \succ C'$ is written if and only if $C \succ_i C'$ for each $i \in C \cap C'$.

Definition 3 *An ordered set of non-single coalitions $(R_1, \ldots, R_J) \subseteq \mathcal{K}$, with $J \geq 3$, is a **ring** if $R_{j+1} \succ R_j$ for $j = 1, \ldots, J$ subscript modulo J.*

For the sake of convenience, we sometimes identify a ring with the non-ordered set of its coalitions, $\mathcal{R} = \{R_1, \ldots, R_J\}$, and refer to coalitions in $\mathcal{R}$ as *ring coalitions*. Notice that the definition of a ring requires that all agents in the intersection of two consecutive ring coalitions should improve. There are several ways to define rings in preferences. Pycia (2012) and Inal (2015) define cyclicity among coalitions by requiring that only one agent at the intersection of two consecutive coalitions strictly prefer the first of them to the second. In both these definitions, unlike ours, other members of two consecutive coalitions can oppose the transition from one coalition to the next.

Definition 4 *An ordered set of coalition structures $(\pi_1, \ldots, \pi_J) \subset \Pi$, with $J \geq 3$, is a **cycle** if $\pi_{j+1} \gg \pi_j$ for $j = 1, \ldots, J$ subscript modulo J.*

Next, we present an algorithm that constructs a ring from a cycle of coalition structures. Let $\mathscr{C} = (\pi_1, \ldots, \pi_J)$ be a cycle of coalition structures, let C_j denote the coalition that is formed in π_j, i.e., $\pi_j \gg \pi_{j-1}$ via C_j, and consider the ordered set $\mathcal{C} = (C_1, \ldots, C_J)$. To construct a ring, proceed as follows:

Algorithm:

Step 1 Set $\overline{R}_1$ as any coalition in $\mathcal{C}$.

Step t Set

$\overline{R}_t \equiv \min_{r\geq 1}\{C_{j+r}$ such that $C_j = \overline{R}_{t-1}$ and $C_j \cap C_{j+r} \neq \emptyset$ with $j + r \bmod J\}$.

IF $\overline{R}_t = \overline{R}_s$ for $s < t$,

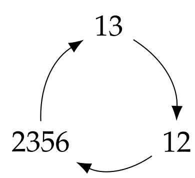
THEN set $(\overline{R}_{s+1}, \ldots, \overline{R}_t)$, and STOP.

ELSE continue to Step $t + 1$.

Notice that in each step of the algorithm a different coalition of $\mathcal{C}$ is selected except in the last step, where only one of the previously selected coalitions is singled out. Therefore, the algorithm stops in at most $J + 1$ steps (recall that $J = |\mathcal{C}|$). The following lemma shows that the ordered set $(\overline{R}_{s+1}, \ldots, \overline{R}_t)$, where s is identified in the algorithm, is actually a ring. To simplify notation, we rename the elements of the ordered set and write $(R_1, \ldots, R_\ell) = (\overline{R}_{s+1}, \ldots, \overline{R}_t)$. The algorithm is illustrated with the following example:

Example 1 *Consider the coalition formation game* $(N, \mathcal{K}, \succ_N)$ *given by the table bellow, where the preferences of the agents are listed in columns in decreasing order. This game has a ring,* $(2356, 13, 12)$*, which is represented in the figure.*[9]

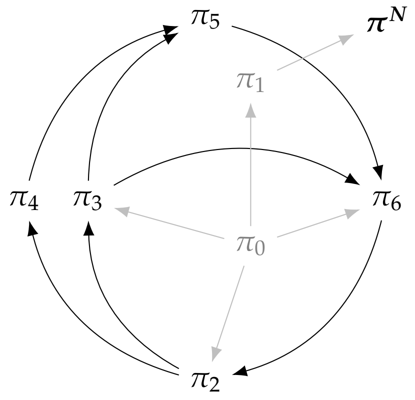
1	**2**	**3**	**4**	**5**	**6**
12	2356	13	**45**	2356	2356
123	**123**	**123**	4	**45**	**6**
13	12	2356		5	
1	2	3			



The associated coalition formation system is represented bellow by a digraph. There is a non-trivial absorbing set $\mathcal{A}^N = \{\pi_2, \pi_3, \pi_4, \pi_5, \pi_6\}$. *The blocking relation* $\gg$

[9] To simplify notation, we omit curly brackets and commas to represent a coalition. For example, coalition $\{1, 2, 3\}$ is simply written as 123.

between coalition structures is represented by arrows. The black ones represent the blocking relation between coalition structures belonging to $\mathcal{A}^N$.



$$\pi_0 = \{1,2,3,4,5,6\}$$
$$\pi_1 = \{123,4,5,6\}$$
$$\pi_2 = \{13,2,4,5,6\}$$
$$\pi_3 = \{12,3,4,5,6\}$$
$$\pi_4 = \{13,2,45,6\}$$
$$\pi_5 = \{12,3,45,6\}$$
$$\pi_6 = \{1,2356,4\}$$
$$\boldsymbol{\pi}^N = \{123,45,6\}$$

Consider cycle $\mathscr{C} = (\pi_2, \pi_4, \pi_5, \pi_6)$. *The set of blocking coalitions between coalition structures is* $C = (13, 45, 12, 2356)$. *Assume that Step 1 of the previous algorithm selects coalition* 45. *The next steps select coalitions* 2356, 13, *and* 12, *respectively. The algorithm ends when coalition* 2356 *is reached again, and ring* $(2356, 13, 12)$ *is obtained.* ◊

Lemma 1 *Let* $\mathscr{C}$ *be a cycle of coalition structures. Then, cycle* $\mathscr{C}$ *induces a ring.*

Proof. Let $\mathscr{C}$ be a cycle of coalition structures. Applying the previous algorithm results in the ordered set $(R_1, \ldots, R_\ell)$. We claim that the ordered set $(R_1, \ldots, R_\ell)$ thus constructed is a ring, i.e. for each R_{j+1} and R_j in the ordered set, $R_{j+1} \succ R_j$ and $\ell \geq 3$. Take any coalition R_j. Coalition R_{j+1} (modulo ℓ) is the closest coalition that has non-empty intersection with R_j (following the modular order of the coalition structures in cycle $\mathscr{C}$), so all the coalition structures between the one in which R_j blocks and the one in which R_{j+1} blocks contain coalition R_j. Let π and π' be the two consecutive coalition structures in $\mathscr{C}$ such that $\pi' \gg \pi$ via R_{j+1}. R_{j+1} is the blocking coalition, so R_{j+1} belongs to π'. Furthermore, since R_j belongs to π and $R_{j+1} \cap R_j \neq \emptyset$, by Definition 1 $R_{j+1} \succ R_j$. Furthermore, $\ell \geq 3$. This holds for the following two facts: (i) there are at least two coalitions in the ordered set, because all the coalitions that block in a cycle are also blocked; (ii) if there are only two coalitions, say R_1 and R_2, then there is an agent $i \in R_1 \cap R_2$ such that $R_1 \succ_i R_2 \succ_i R_1$, which by transitivity implies $R_1 \succ_i R_1$, a contradiction. □

The following theorem, which plays a central role in our characterization result, establishes the relationship between a ring of coalitions in the preference profile and a cycle of coalition structures of the associated coalition formation system.

Theorem 1 *A coalition formation game has a ring of coalitions if and only if its associated coalition formation system has a cycle of coalition structures.*

Proof. ($\Longleftarrow$) This is proven by Lemma 1.
($\Longrightarrow$) Let $(R_1, \ldots, R_J)$ be a ring in the coalition formation game $(N, \mathcal{K}, \succ_N)$. This ring induces a cycle of coalition structures $\mathscr{C} = (\pi_1, \ldots, \pi_J)$ where π_j is defined as follows:

$$\pi_j(i) = \begin{cases} R_j & \text{for } i \in R_j \\ \{i\} & \text{otherwise.} \end{cases}$$

Note that π_j is obtained from π_{j-1} by satisfying blocking coalition R_j for each $j = 1, \ldots, J$. □

4 Effective ring component and characterization

In this section we characterize a non-trivial absorbing set in terms of effective ring components. Subsection 4.1 defines the notion of effective ring component and illustrates it with two numerical examples. Subsection 4.2 contains the characterization result illustrated with two numerical examples.

4.1 Effective ring component

A coalition formation game may have multiple rings, some of which may overlap. A collection of *overlapping rings* is a set of rings such that for each $\mathcal{R}$ in the collection there is another $\mathcal{R}'$ in the collection such that $\mathcal{R} \cap \mathcal{R}' \neq \emptyset$. However, not all rings of a coalition formation game induce a non-trivial absorbing set and attention must be paid to those that do.

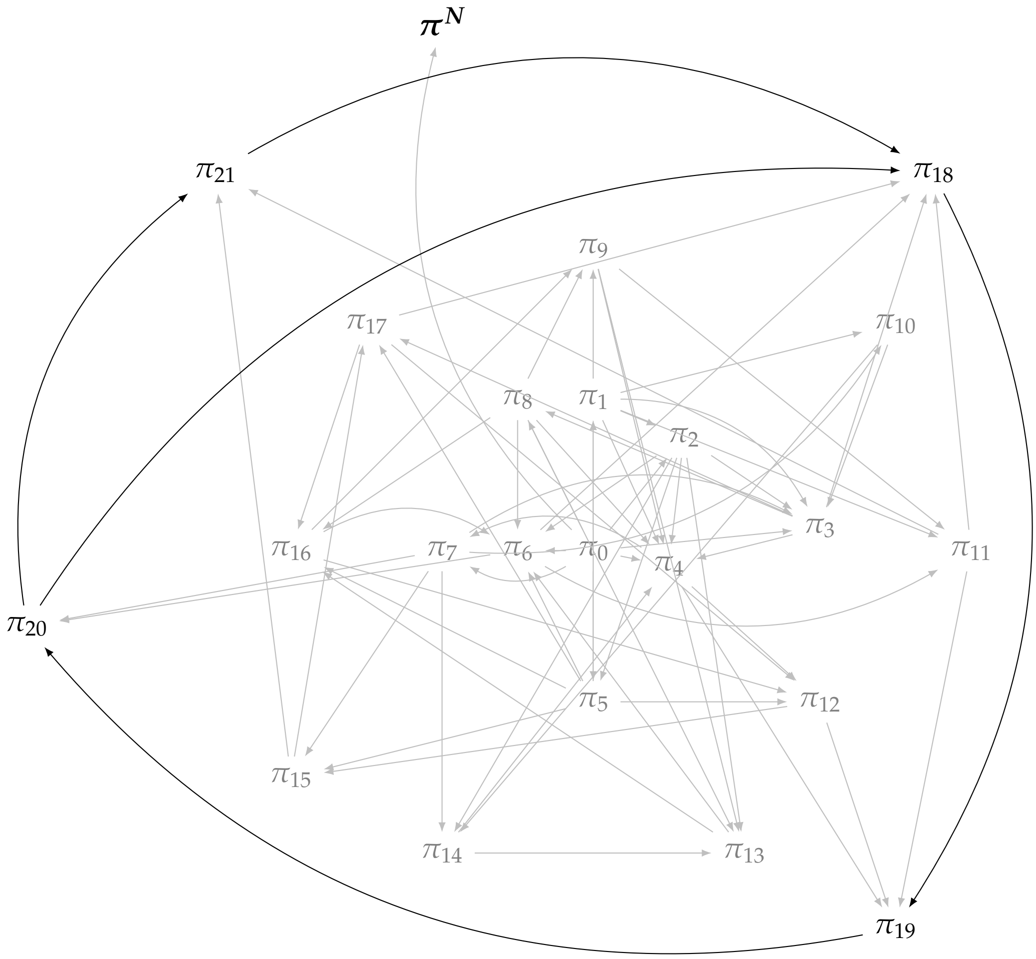
Definition 5 *Let $(N, \mathcal{K}, \succ_N)$ be a coalition formation game. A union of overlapping rings is a* ***ring component*** $\boldsymbol{\mathcal{RC}}$ *if there is a non-trivial absorbing set $\mathcal{A}^N$ such that for each ring coalition $R \in \mathcal{RC}$, there are $\pi, \pi' \in \mathcal{A}^N$ with $\pi' \gg \pi$ via R. In this case, $\mathcal{RC}$ is said to be* ***embedded in*** $\boldsymbol{\mathcal{A}^N}$.

Notice that for each pair of different coalitions $C, C' \in \mathcal{RC}$ there is a finite sequence of coalitions $C = C_0, C_1 \ldots, C_{J-1}, C_J = C'$ that belong to $\mathcal{RC}$ such that $C_J \succ C_{J-1} \succ \ldots \succ C_1 \succ C_0$. Example 2 illustrates a coalition formation game with several rings some of which are not embedded in its non-trivial absorbing set. Example 3 illustrates a coalition formation game with two overlapping rings embedded in its non-trivial absorbing set, that justifies the previous definition.

Example 2 *Consider the coalition formation game $(N, \mathcal{K}, \succ_N)$ given by the following table:*

1	**2**	**3**	**4**	**5**	**6**	**7**
15	26	13	456	457	***N***	***N***
14	23	***N***	***N***	***N***	26	457
12	***N***	23	14	15	456	7
N	12	3	457	456	6	
13	2		4	5		
1						

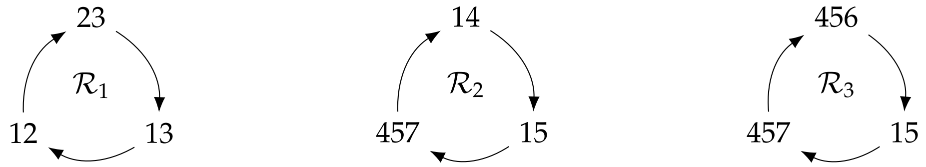
The associated coalition formation system can be represented by the following digraph:



where the coalition structures are

$$\begin{array}{lll}
\pi_0 = \{1,2,3,4,5,6,7\} & \pi_8 = \{1,2,3,456,7\} & \pi_{16} = \{1,23,456,7\} \\
\pi_1 = \{13,2,4,5,6,7\} & \pi_9 = \{13,2,456,7\} & \pi_{17} = \{14,23,5,6,7\} \\
\pi_2 = \{12,3,4,5,6,7\} & \pi_{10} = \{13,2,457,6\} & \pi_{18} = \{14,26,3,5,7\} \\
\pi_3 = \{14,2,3,5,6,7\} & \pi_{11} = \{13,26,4,5,7\} & \pi_{19} = \{15,26,3,4,7\} \\
\pi_4 = \{15,2,3,4,6,7\} & \pi_{12} = \{15,23,4,6,7\} & \pi_{20} = \{1,26,457,3\} \\
\pi_5 = \{1,23,4,5,6,7\} & \pi_{13} = \{12,3,456,7\} & \pi_{21} = \{13,26,457\} \\
\pi_6 = \{1,26,3,4,5,7\} & \pi_{14} = \{12,3,457,6\} & \boldsymbol{\pi}^N = \{N\} \\
\pi_7 = \{1,2,3,457,6\} & \pi_{15} = \{1,23,457,6\} &
\end{array}$$

This game has $\{N\}$ *as the stable coalition structure and* $\mathcal{A}^N = \{\pi_{18}, \pi_{19}, \pi_{20}, \pi_{21}\}$ *as the non-trivial absorbing set.*

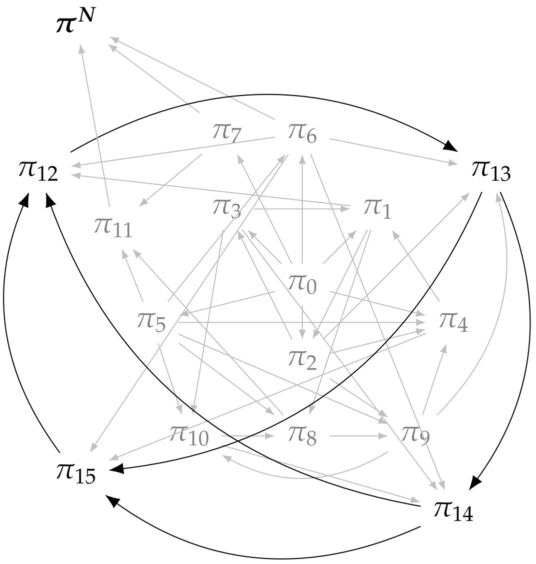


The game has three rings: $\mathcal{R}_1 = \{12, 23, 13\}, \mathcal{R}_2 = \{457, 14, 15\}$, *and* $\mathcal{R}_3 = \{457, 456, 15\}$. *Observe that* $\mathcal{R}_1$ *is not embedded in* $\mathcal{A}^N$, *and even though* $\mathcal{R}_2$ *and* $\mathcal{R}_3$ *overlap only* $\mathcal{R}_2$ *is a ring component embedded in* $\mathcal{A}^N$. $\Diamond$

Example 3 *Consider the coalition formation game* $(N, \mathcal{K}, \succ_N)$ *given by the following table:*

1	**2**	**3**	**4**	**5**	**6**
12	234	13	234	**56**	**56**
123	23	**123**	45	45	6
13	**123**	23	**4**	5	
1	12	234			
	2	3			

The coalition structures and the digraph of the associated coalition formation system are the following:



$\pi_0 = \{1, 2, 3, 4, 5, 6\}$ $\quad \pi_1 = \{13, 2, 4, 5, 6\}$

$\pi_2 = \{12, 3, 4, 5, 6\}$ $\quad \pi_3 = \{23, 1, 4, 5, 6\}$

$\pi_4 = \{234, 1, 5, 6\}$ $\quad \pi_5 = \{45, 1, 2, 3, 6\}$

$\pi_6 = \{56, 1, 2, 3, 4\}$ $\quad \pi_7 = \{123, 4, 5, 6\}$

$\pi_8 = \{13, 2, 45, 6\}$ $\quad \pi_9 = \{12, 3, 45, 6\}$

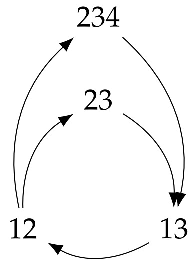
$\pi_{10} = \{23, 1, 45, 6\}$ $\quad \pi_{11} = \{123, 45, 6\}$

$\pi_{12} = \{13, 2, 4, 56\}$ $\quad \pi_{13} = \{12, 3, 4, 56\}$

$\pi_{14} = \{23, 1, 4, 56\}$ $\quad \pi_{15} = \{234, 1, 56\}$

$\boldsymbol{\pi}^N = \{123, 4, 56\}$

This game has $\pi^N = \{123, 4, 56\}$ as its stable coalition structure and $\mathcal{A}^N = \{\pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}\}$ as the non-trivial absorbing set. The game has two rings: $\mathcal{R}_1 = \{12, 23, 13\}$ and $\mathcal{R}_2 = \{13, 12, 234\}$. $\mathcal{R}_1$ and $\mathcal{R}_2$ are overlapping rings, and their union is the only ring component embedded in $\mathcal{A}^N$.



$\diamondsuit$

Not every ring component embedded in a non-trivial absorbing set is responsible of the lack of convergence to stability. A ring component without "external" blocking is found to do the job. Before we introduce this requirement some notation needs to be added. Given an absorbing set $\mathcal{A}^N$, let $\mathcal{C}(\mathcal{A}^N)$ be the set of non-single coalitions participating in $\mathcal{A}^N$. Formally, $\mathcal{C}(\mathcal{A}^N) = \{C \in \mathcal{K} : |C| > 1$ and there is $\pi \in \mathcal{A}^N$ such that $C \in \pi\}$.

Definition 6 *Let $(N, \mathcal{K}, \succ_N)$ be a coalition formation game, $\mathcal{A}^N$ a non-trivial absorbing set, and $\mathcal{RC}$ a ring component embedded in $\mathcal{A}^N$. Coalition $X \in \mathcal{C}(\mathcal{A}^N) \setminus \mathcal{RC}$ is an **exit of $\mathcal{RC}$ in $\mathcal{A}^N$** if there are $\pi, \pi' \in \mathcal{A}^N$ and $R \in \mathcal{RC}$ such that:*

(i) $R \in \pi$,

(ii) $X \succ R$, *and*

(iii) $\pi' \gg \pi$ *via* X.

This definition will be illustrated in Example 4. Notice that an exit of a ring component could be a coalition of another ring component. Now, we are in the position to introduce the notion used to characterize a non-trivial absorbing set.

Definition 7 *Let $(N, \mathcal{K}, \succ_N)$ be a coalition formation game. A ring component $\mathcal{RC}$ is **effective** if there is a non-trivial absorbing set $\mathcal{A}^N$ such that $\mathcal{RC}$ is embedded and has no exit in $\mathcal{A}^N$.*

4.2 The characterization result

To present the characterization result, we first show that it is possible to recover the collection of ring components embedded in a non-trivial absorbing set.

Proposition 2 *A non-trivial absorbing set of a coalition formation system induces a collection of ring components.*

Proof. Let $\mathcal{A}^N$ be a non-trivial absorbing set. Notice that, given any two different coalition structures in $\mathcal{A}^N$, by Definition 2 there is a cycle of coalition structures in $\mathcal{A}^N$ that includes those structures. $\mathcal{A}^N$ can therefore be seen as the union of all such cycles. Thus, for each cycle of coalition structures in $\mathcal{A}^N$, the algorithm developed in Section 3 constructs a ring. By merging overlapping rings, all ring components embedded in $\mathcal{A}^N$ can be constructed. $\square$

The existence of a collection of ring components in a non-trivial absorbing set suggests that the relation between them should be analyzed. That relation is defined by using the notion of a path of coalitions within a non-trivial absorbing set.

Definition 8 *Let $\mathcal{A}^N$ be a non-trivial absorbing set and let C, C' be two different coalitions in $\mathcal{C}(\mathcal{A}^N)$. There is a* ***path from C to C' in $\mathcal{A}^N$*** *if, for each $j = 0, \ldots, t$, there are $\pi_j \in \mathcal{A}^N$ and $X_j \in \pi_j$ such that, for each $j = 0, \ldots, t-1$,*

(i) $\pi_{j+1} \gg \pi_j$,

(ii) $X_0 = C$, $X_t = C'$, *and* $X_{j+1} = X_j$ *or* $X_{j+1} \succ X_j$.

Observe that Condition (i) requires the blocking relation between any two consecutive coalition structures of the path; and Condition (ii) requires that every pair of consecutive blocking coalitions in the path intersect each other. Notice that whenever $X_{j+1} \neq X_j$, $\pi_{j+1} \gg \pi_j$ via X_{j+1}.

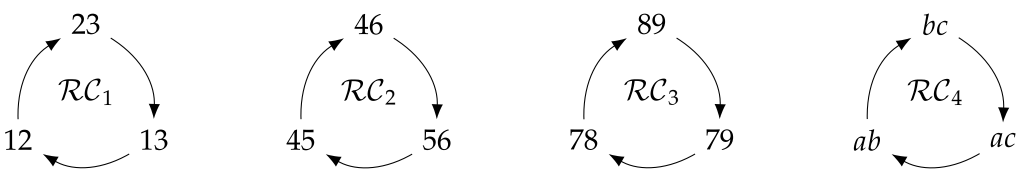
Definition 9 *Let $\mathcal{A}^N$ be a non-trivial absorbing set and let $\mathcal{RC}$ and $\mathcal{RC}'$ be two different ring components embedded in $\mathcal{A}^N$. Define* $\boldsymbol{\mathcal{RC} \triangleleft \mathcal{RC}'}$ *if and only if there are coalitions $C \in \mathcal{RC}$ and $C' \in \mathcal{RC}'$ such that there is a path from C to C' in $\mathcal{A}^N$.*

The example below illustrates these two definitions.

Example 4 *Consider the coalition formation game $(N, \mathcal{K}, \succ_N)$ given by the following table:*

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	***a***	***b***	***c***
N	**N**	**N**	47	56	46	**N**	**N**	**N**	*ab*	*bc*	*ac*
14	23	13	14	**N**	**N**	7*a*	89	79	**N**	**N**	**N**
12	13	23	45	45	56	47	78	89	*ac*	*ab*	*bc*
13	2	3	**N**	5	6	78	8	9	7*a*	*b*	*c*
1			46			79			*a*		
			4			7					

In this game, the stable coalition structure is $\pi^N = \{N\}$. The non-trivial absorbing set $\mathcal{A}^N$ is formed by multiple overlapping cycles of coalition structures. Each coalition structure in $\mathcal{A}^N$ contains coalition ab, bc or ac, while the remaining agents are either grouped in two-agent coalitions or are singletons. If the algorithm in Section 3 is applied to each of the cycles in $\mathcal{A}^N$, it is possible to construct the four rings components embedded in $\mathcal{A}^N$: $\mathcal{RC}_1 = \{12, 23, 13\}$, $\mathcal{RC}_2 = \{45, 46, 56\}$, $\mathcal{RC}_3 = \{78, 89, 79\}$, and $\mathcal{RC}_4 = \{ab, bc, ac\}$.



Notice that coalitions $14, 47, 7a \in \mathcal{C}(\mathcal{A}^N)$ are disregarded by the algorithm, i.e. although these coalitions block some coalition structures of $\mathcal{A}^N$ they do not belong to any ring component. Observe that coalition 14 is an exit of $\mathcal{RC}_1$, coalitions 14 and 47 are exits of $\mathcal{RC}_2$, and coalitions 47 and 7a are exits of $\mathcal{RC}_3$. Consider the following sequence of coalition structures π_j in $\mathcal{A}^N$ and its blocking coalitions X_j:

$$\pi_0 = \{\mathbf{12}, 3, 4, 56, 78, 9, ab, c\} \qquad X_0 = 12$$
$$\pi_1 = \{\mathbf{14}, 2, 3, 56, 78, 9, ab, c\} \qquad X_1 = 14$$
$$\pi_2 = \{\mathbf{14}, 2, 3, 56, 7, 89, ab, c\} \qquad X_2 = 14$$
$$\pi_3 = \{1, 2, 3, \mathbf{47}, 56, 89, ab, c\} \qquad X_3 = 47$$
$$\pi_4 = \{1, 2, 3, \mathbf{47}, 56, 89, a, bc\} \qquad X_4 = 47$$
$$\pi_5 = \{1, 2, 3, 4, 56, \mathbf{7a}, 89, bc\} \qquad X_5 = 7a$$
$$\pi_6 = \{1, 2, 3, 4, 56, 7, 89, \mathbf{ac}, b\} \qquad X_6 = ac$$

This sequence fulfills the conditions of Definition 8, so there is a path from coalition 12 *of $\mathcal{RC}_1$ to coalition ac of $\mathcal{RC}_4$, which means that $\mathcal{RC}_1 \lhd \mathcal{RC}_4$.* $\Diamond$

Let $\lhd^T$ be the transitive closure of $\lhd$. Next, we show some properties of this relation.

Lemma 2 *Relation $\lhd^T$ is a strict partial order.*

Proof. To prove that $\lhd^T$ is a strict partial order, it must be shonw that it is a transitive, irreflexive relation. By definition, transitivity holds. To show irreflexivity of $\lhd^T$ it suffices to prove the acyclicity of $\lhd$, since this implies asymmetry of $\lhd^T$ and, in turn, irreflexivity of $\lhd^T$. Assume then that $\lhd$ is not acyclic. This implies that there are ring components $\mathcal{RC}_1, \ldots, \mathcal{RC}_r$ with $r \geq 3$ embedded in $\mathcal{A}^N$ such that $\mathcal{RC}_j \lhd \mathcal{RC}_{j-1}$ for $j \in \{2, \ldots, r\}$ and $\mathcal{RC}_1 \lhd \mathcal{RC}_r$. This in turn implies that there is a cycle of coalition structures in $\mathcal{A}^N$ that generates a ring containing coalitions of these ring components. This contradicts the definition of ring component. Thus, $\lhd$ is acyclic and therefore $\lhd^T$ is a strict partial order. $\square$

Relation $\lhd^T$ enables us to link the ring components, establishing a sort of hierarchy among them, until a maximal one is found. This maximal ring component happens to be effective. To prove this, in the following lemma we show that there is at least one ring component which is maximal for $\lhd^T$.[10]

[10] Recall that an element is *maximal* for a strict partial order if it is not smaller than any other element in the set.

Lemma 3 *Let* $(N, \mathcal{K}, \succ_N)$ *be a coalition formation game. If there is a non-trivial absorbing set* $\mathcal{A}^N$*, then there is a ring component embedded in* $\mathcal{A}^N$ *which is maximal for* $\lhd^T$*.*

Proof. Without loss of generality, let $\mathcal{RC}_1, \ldots, \mathcal{RC}_s$ be the finite list of ring components embedded in $\mathcal{A}^N$. Define $\mathcal{RC}_1^\star = \mathcal{RC}_1$ and, for $j \in \{2, \ldots, s\}$, set

$$\mathcal{RC}_j^\star = \begin{cases} \mathcal{RC}_j & \text{if } \mathcal{RC}_{j-1} \lhd^T \mathcal{RC}_j \\ \mathcal{RC}_{j-1}^\star & \text{otherwise} \end{cases}$$

Thus, by construction, and since $\lhd^T$ is a strict partial order by Lemma 2, $\mathcal{RC}_s^\star$ is maximal for $\lhd^T$. □

The following lemma shows that if a ring component embedded in a non-trivial absorbing set has an exit in that absorbing set, it is not maximal for $\lhd^T$.

Lemma 4 *Let* $\mathcal{A}^N$ *be a non-trivial absorbing set and let* $\mathcal{RC}$ *be a ring component embedded in* $\mathcal{A}^N$*. If there is an exit of* $\mathcal{RC}$ *in* $\mathcal{A}^N$*, then there is a ring component* $\mathcal{RC}'$ *embedded in* $\mathcal{A}^N$ *such that* $\mathcal{RC} \lhd \mathcal{RC}'$*.*

Proof. Let $\mathcal{A}^N$ be a non-trivial absorbing set and let $\mathcal{RC}$ be a ring component embedded in $\mathcal{A}^N$. Assume X is an exit of $\mathcal{RC}$ in $\mathcal{A}^N$. Therefore, there are $\tilde{\pi}, \pi^\star \in \mathcal{A}^N$ and $R \in \mathcal{RC}$ such that $R \in \tilde{\pi}$, $X \succ R$, and $\pi^\star \gg \tilde{\pi}$ via X. If X belongs to a ring component $\mathcal{RC}'$ embedded in $\mathcal{A}^N$, then $\mathcal{RC} \lhd \mathcal{RC}'$. Assume, next, that X is not part of any ring component embedded in $\mathcal{A}^N$. Define the set

$$\mathscr{B} = \{Y \in \mathcal{C}(\mathcal{A}^N) : \text{ there is a path from } X \text{ to } Y \text{ in } \mathcal{A}^N\}$$

and let $\mathscr{R}$ denote the collection of all coalitions in ring components embedded in $\mathcal{A}^N$. We claim that $\mathscr{B} \cap \mathscr{R} \neq \emptyset$. To see this, take any $C_1 \in \mathscr{B}$. This implies that there is $\pi_1 \in \mathcal{A}^N$ such that $C_1 \in \pi_1$ and $\pi_1 \gg^T \pi^\star$. Starting from partition π_1, "move" within the absorbing set until a partition π_2 is reached in which C_1 is no longer present. Let C_2 be such that $C_2 \in \pi_2$ and $C_2 \succ C_1$. Notice that $C_2 \in \mathscr{B}$ as well. Proceeding in the same way, it is possible to construct a sequence in $\mathscr{B}$ such that $C_{t+1} \succ C_t$ for each t. Since $\mathscr{B}$ is finite, there is $t^\star$ such that $C_{t^\star} = C_{t'}$ for $t' < t^\star$. Without loss of generality, we can choose $t^\star$ to be the smallest that fulfills this property. This implies that coalitions $C_{t'+1}, C_{t'+2}, \ldots, C_{t^\star}$ form a ring, and therefore are in $\mathscr{R}$. Hence, $\mathscr{B} \cap \mathscr{R} \neq \emptyset$. Thus, there are $R' \in \mathscr{B} \cap \mathscr{R}$ and a ring component $\mathcal{RC}'$ such that $R' \in \mathcal{RC}'$. Therefore, there is a path from X to R' and, consequently, $\mathcal{RC} \lhd \mathcal{RC}'$. □

Lemmata 3 and 4 together with Proposition 2 make it possible to characterize stable coalition formation games in terms of effective ring components.

Theorem 2 *A stable coalition formation game lacks convergence to stability if and only if it has an effective ring component.*

Proof. By Proposition 1, it suffices to prove that there is a non-trivial absorbing set if and only if there is an effective ring component.

($\Longrightarrow$) Let $\mathcal{A}^N$ be a non-trivial absorbing set. By Proposition 2, all ring components embedded in $\mathcal{A}^N$ can be constructed. By Lemma 3, one of them is maximal for $\lhd^T$, say, $\mathcal{RC}^\star$. Assume that $\mathcal{RC}^\star$ has an exit in $\mathcal{A}^N$. Then, by Lemma 4, there is a ring component $\mathcal{RC}'$ such that $\mathcal{RC}^\star \lhd \mathcal{RC}'$. This contradicts the maximality of $\mathcal{RC}^\star$ for $\lhd^T$. Therefore, $\mathcal{RC}^\star$ has no exit in $\mathcal{A}^N$, and it is an effective ring component for $\mathcal{A}^N$.

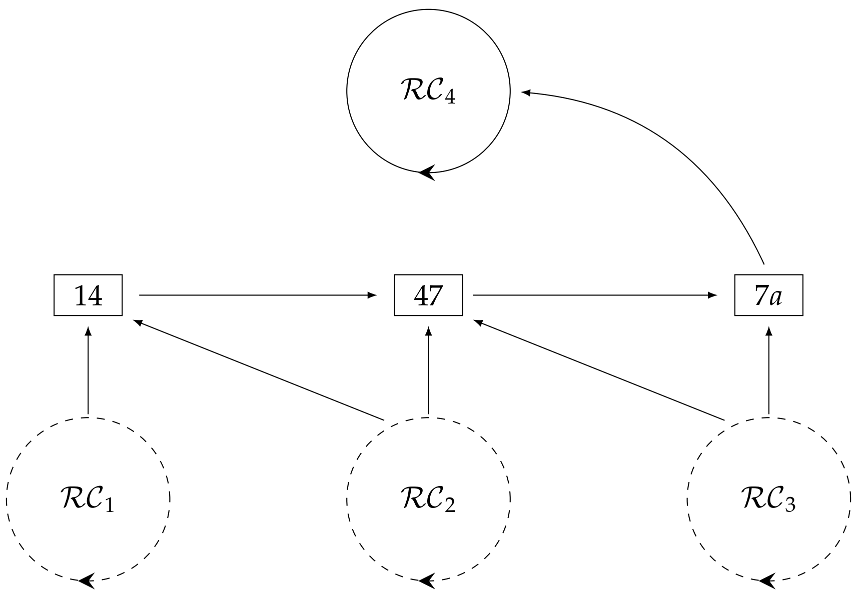
($\Longleftarrow$) Let $\mathcal{RC}$ be an effective ring component. Then, by the definition of ring component, there is a non-trivial absorbing set $\mathcal{A}^N$. □

The following corollary follows immediately from the characterization result.

Corollary 1 *A stable coalition formation game without rings in preferences exhibits convergence to stability.*

Observe that the preferences over coalitions of the agents of the effective ring components, unlike those of the remaining agents, are responsible for the existence of a non-trivial absorbing set and, as a result, for the lack of convergence to stability. Example 4 illustrates how the non-effective ring components are related to the effective one.

Example 4 (continued) *Recall that this game has four disjoint rings embedded in* $\mathcal{A}^N$ *and therefore ring components:* $\mathcal{RC}_1 = \{12, 23, 13\}$, $\mathcal{RC}_2 = \{45, 46, 56\}$, $\mathcal{RC}_3 = \{78, 89, 79\}$, *and* $\mathcal{RC}_4 = \{ab, bc, ac\}$. *Furthermore, coalition* 14 *is an exit of* $\mathcal{RC}_1$, *coalitions* 14 *and* 47 *are exits of* $\mathcal{RC}_2$, *and coalitions* 47 *and* 7*a* *are exits of* $\mathcal{RC}_3$. *Hence, none of these ring components are effective, while ring component* $\mathcal{RC}_4$ *has no exit and is therefore effective. The coalitions disregarded by the algorithm* 14, 47 *and* 7*a* *connect the ring components within* $\mathcal{A}^N$ *so that* $\mathcal{RC}_1 \lhd^T \mathcal{RC}_4$, $\mathcal{RC}_2 \lhd^T \mathcal{RC}_4$ *and* $\mathcal{RC}_3 \lhd^T \mathcal{RC}_4$. *These relations are illustrated by the following figure:*



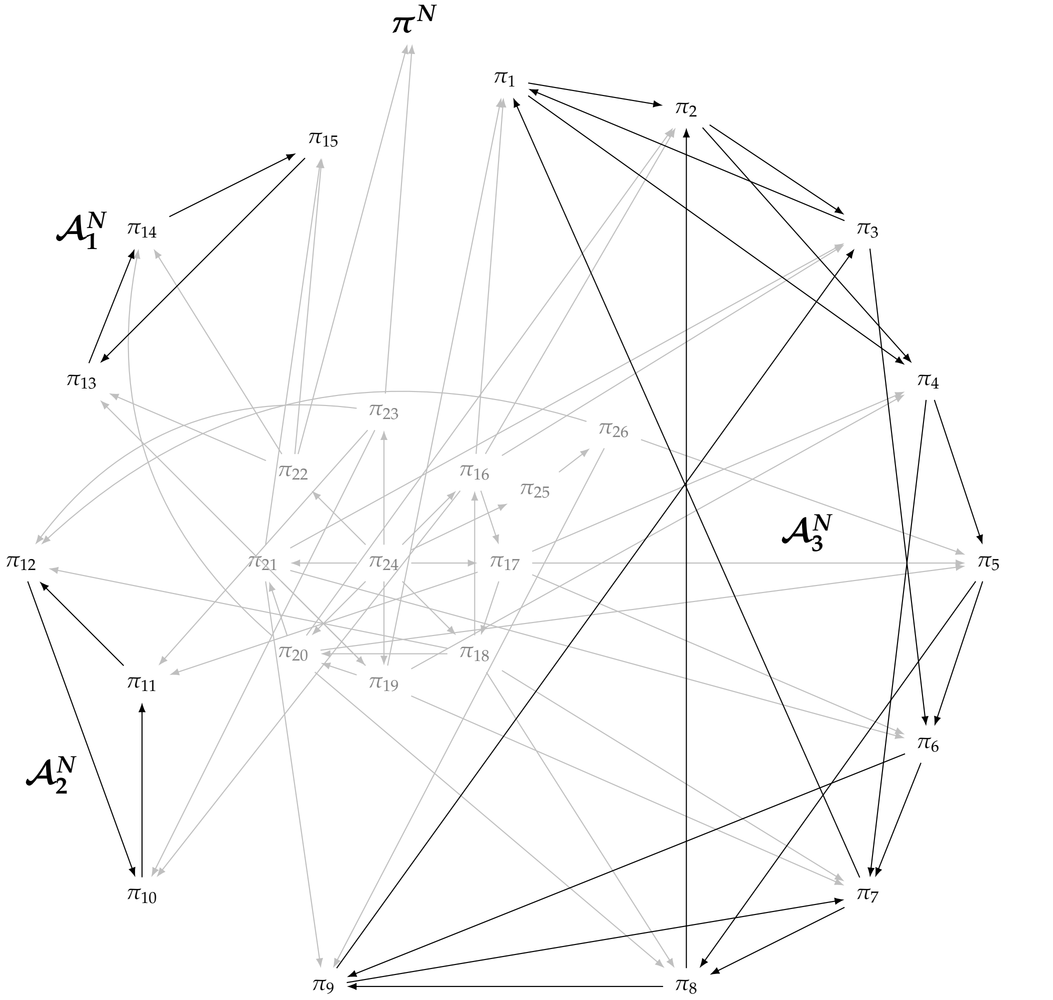
Notice that the set of agents $\{a, b, c\}$ *is responsible for the existence of the non-trivial absorbing set* $\mathcal{A}^N$ *so that if ring coalitions* ab, bc *and* ac *become non-permissible, it is easy to see that there is convergence to* π^N. $\Diamond$

Lastly, recall that only stable coalition formation games are considered here. However, since the analysis cover the agents' preferences that induce a non-trivial absorbing set, the number of stable coalition structures that a game may have is irrelevant in obtaining the characterization result. To conclude this section, we present an example that shows that a coalition formation games may have multiple non-trivial absorbing sets.

Example 5 *Consider the coalition formation game* $(N, \mathcal{K}, \succ_N)$ *given by the following table:*

1	**2**	**3**	**4**	**5**	**6**	**7**
167	23	13	456	456	67	457
12	**123**	**123**	**4567**	**4567**	**4567**	**4567**
123	12	23	457	457	456	67
13	2	3	4	5	167	167
1					6	7

The associated coalition formation system can be represented by the following digraph:



where the coalition structures are

$\pi_1 = \{13, 2, 457, 6\}$	$\pi_8 = \{12, 3, 4, 5, 67\}$	$\pi_{15} = \{123, 4, 5, 67\}$	$\pi_{21} = \{1, 23, 4, 5, 6, 7\}$
$\pi_2 = \{12, 3, 457, 6\}$	$\pi_9 = \{1, 23, 4, 5, 67\}$	$\pi_{16} = \{1, 2, 3, 4, 5, 67\}$	$\pi_{22} = \{12, 3, 4, 5, 6, 7\}$
$\pi_3 = \{1, 23, 457, 6\}$	$\pi_{10} = \{13, 2, 4567\}$	$\pi_{17} = \{1, 2, 3, 456, 7\}$	$\pi_{23} = \{13, 2, 4, 5, 6, 7\}$
$\pi_4 = \{13, 2, 456, 7\}$	$\pi_{11} = \{12, 3, 4567\}$	$\pi_{18} = \{1, 2, 3, 457, 6\}$	$\pi_{24} = \{1, 2, 3, 4, 5, 6, 7\}$
$\pi_5 = \{12, 3, 456, 7\}$	$\pi_{12} = \{1, 23, 4567\}$	$\pi_{19} = \{123, 4, 5, 6, 7\}$	$\pi_{25} = \{1, 2, 3, 4567\}$
$\pi_6 = \{1, 23, 456, 7\}$	$\pi_{13} = \{123, 457, 6\}$	$\pi_{20} = \{167, 2, 3, 4, 5\}$	$\pi_{26} = \{167, 2, 3, 4, 5\}$
$\pi_7 = \{13, 2, 4, 5, 67, \}$	$\pi_{14} = \{123, 456, 7\}$	$\pi^N = \{123, 4567\}$	

This game has $\pi^N = \{123, 4567\}$ as its only stable coalition structure and three non-trivial absorbing sets of the associated coalition formation system. There are two disjoint rings: $\mathcal{R}_1 = \{12, 23, 13\}$ and $\mathcal{R}_2 = \{457, 456, 67\}$.

Ring $\mathcal{R}_1$ is the only (effective) ring component embedded in the absorbing set $\mathcal{A}_1^N$. Ring $\mathcal{R}_2$ is the only (effective) ring component embedded in the absorbing set $\mathcal{A}_2^N$. To see this, consider either $\mathcal{R}_1$ with the stable coalition 4567 or $\mathcal{R}_2$ with the stable coalition 123. However, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are embedded in $\mathcal{A}_3^N$ and have no exit, so both are effective.

◊

5 Coalition formation games and sharing rules

As mentioned in the Introduction, the configuration of coalition formation games may depend on how the output produced by each coalition is distributed among its members. Indeed, the sharing rule chosen to divide up each coalitional output is crucial for the existence of stability and convergence to stability. Subsections 5.1 and 5.2 analyze whether the main sharing rules considered in Pycia (2012) and Gallo and Inarra (2018) induce coalition formation games that exhibit convergence to stability.

5.1 Coalition formation games and bargaining solutions

Pycia (2012) presents a model in which there is a set of agents, each endowed with a utility function, who form coalitions that produce outputs to be distributed among its members. He shows that under a rich domain of preferences and some restrictions on coalitions there is a stable coalition structure for each preference profile if and only if agents' preferences satisfy pairwise alignment. Agents' preferences are pairwise aligned if any two agents rank coalitions that contain both of them in the same way. Formally, in our setting of strict preferences, a preference profile $\succ_N$ over coalitions is *pairwise aligned* if for all $i, j \in C \cap C'$ it holds that $C \succ_i C'$ if and only if $C \succ_j C'$.

Given a set of agents N and a set of coalitions $\mathcal{K} \subseteq 2^N \setminus \{\emptyset\}$, a *coalitional bargaining problem* is a tuple $(U_N, y(C)_{C\in\mathcal{K}})$ where $U_N = (U_i)_{i\in N}$ is a vector of utility functions $U_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and, for each $C \in \mathcal{K}$, $y(C)$ is the output produced by coalition C. When agent $i \in C$ gets the share x of output $y(C)$ her utility gives her $U_i(x)$. Given $C \in \mathcal{K}$, the *bargaining problem for* C is $(U_C, y(C))$ where $U_C = (U_i)_{i\in C}$ is the utility vector of agents in C and $y(C)$ is the output of coalition C.[11] An *allocation* for the bargaining problem for C, is a vector $x = (x_i)_{i\in C} \in \mathbb{R}^C_+$ such that $\sum_{i\in C} x_i = y(C)$. A *bargaining rule* is a mapping that associates an allocation with each bargaining problem.

Given a coalitional bargaining problem $(U_N, y(C)_{C\in\mathcal{K}})$, a bargaining rule F induces a coalition formation game $(N, \mathcal{K}, \succ_N)$ in the following way: for each $i \in N$ and each pair $C, C' \in \mathcal{K}$ with $i \in C \cap C'$, if $F_i(U_C, y(C)) > F_i(U_{C'}, y(C'))$ then $C \succ_i C'$. Note that for the game to be well-defined, no pair of bargaining problems should allocate the same amount to agent i. A bargaining rule is *pairwise aligned* if the coalition formation game induced is pairwise aligned for each bargaining problem.

Theorem 3 *Any coalition formation game induced by a pairwise aligned bargaining rule exhibits convergence to stability.*

[11] We normalize all bargaining problems so that the disagreement point is equal to the origin.

Proof. Let $(N, \mathcal{K}, \succ_N)$ be a coalition formation game induced by a pairwise aligned bargaining rule. Pycia (2012) guarantees that $(N, \mathcal{K}, \succ_N)$ is a stable coalition formation game with no rings.[12] By Corollary 1, $(N, \mathcal{K}, \succ_N)$ exhibits convergence to stability. □

Unlike the Kalai-Smorondinsky bargaining rule (Kalai and Smorodinsky, 1975), the Nash bargaining rule (Nash, 1950) is included in the class of pairwise aligned bargaining rules (see Pycia, 2012, p.331) and therefore guarantees stability. However, even if one considers only stable coalition formation games induced by the Kalai-Smorodinsky solution, it is found that they may lack convergence to stability. Below, we define these two rules and illustrate their behavior when inducing coalition formation games. Given $C \in \mathcal{K}$, the Nash bargaining rule for problem $(U_C, y(C))$ is determined by solving:

$$\max_{x_i \geq 0} \prod_{i \in C} U_i(x) \text{ subject to } \sum_{i \in C} x_i = y(C).$$

Given $C \in \mathcal{K}$, the Kalai-Smorodinsky bargaining rule for problem $(U_C, y(C))$ is determined by solving:

$$\frac{U_i(x_i)}{U_i(y(C))} = \frac{U_j(x_j)}{U_j(y(C))} \text{ for all } i, j \in C \text{ subject to } \textstyle\sum_{i \in C} x_i = y(C).$$

Example 6 *Consider a risk-averse firm f and a risk-neutral firm g that can employ either one or two risk-averse workers w_1, w_2 whose utilities are given by*

$$U_f(x) = x^{1/4},\ U_g = x,\ U_{w_1}(x) = x^{1/6},\ U_{w_2}(x) = x^{1/2}.$$

The following table provides the coalitions and the allocation given by the Nash and the Kalai-Smorodinsky (K-S) bargaining solutions for different levels of outputs:

[12] Pycia (2012) shows that each pairwise aligned bargaining rule induces a stable coalition formation game (Corollary 1 in Pycia (2012)) with a rich domain of preferences. Lemmata 3 and 4 in Pycia (2012) state that a coalition formation game with rich domain and pairwise aligned preferences has no "n-cycles in preferences". The non-existence of "n-cycles in preferences" in his setting implies the non-existence of rings in our setting.

Coalitions	$f\ w_1w_2$	$g\ w_1w_2$	$f\ w_1$	$f\ w_2$	$g\ w_1$	$g\ w_2$
Outputs	43	83	20	37	1	1
Nash	(11.7, 7.8, 23.5)	(49.8, 8.3, 24.9)	(12, 8)	(12.3, 24.7)	(0.8, 0.2)	(0.7, 0.3)
K-S	(12.7, 6.9, 23.4)	(49.6, 3.8, 29.6)	(11.4, 8.6)	(14.1, 22.9)	(0.8, 0.2)	(0.6, 0.4)

The coalition formation game induced by Nash bargaining is:

$\boldsymbol{f}$	$\boldsymbol{g}$	$\boldsymbol{w_1}$	$\boldsymbol{w_2}$
$f\ w_2$	$\boldsymbol{g\ w_1w_2}$	$\boldsymbol{g\ w_1w_2}$	$\boldsymbol{g\ w_1w_2}$
$f\ w_1$	$g\ w_1$	$f\ w_1$	$f\ w_2$
$f\ w_1w_2$	$g\ w_2$	$f\ w_1w_2$	$f\ w_1w_2$
$\boldsymbol{f}$	g	$g\ w_1$	$g\ w_2$
		w_1	w_2

Observe that the stable coalition structure is $\pi^N = \{f, g\ w_1w_2\}$, the preference profile is pairwise aligned and there are no rings in preferences. Therefore, the game induced by Nash bargaining exhibits convergence to stability.

The coalition formation game induced by Kalai-Smorodinsky bargaining is:

$\boldsymbol{f}$	$\boldsymbol{g}$	$\boldsymbol{w_1}$	$\boldsymbol{w_2}$
$f\ w_2$	$g\ w_1w_2$	$f\ w_1$	$g\ w_1w_2$
$\boldsymbol{f\ w_1w_2}$	$g\ w_1$	$\boldsymbol{f\ w_1w_2}$	$\boldsymbol{f\ w_1w_2}$
$f\ w_1$	$g\ w_2$	$g\ w_1w_2$	$f\ w_2$
f	$\boldsymbol{g}$	$g\ w_1$	$g\ w_2$
		w_1	w_2

Observe that the stable coalition structure is $\pi^N = \{f\ w_1w_2, g\}$, the preference profile is not pairwise aligned and there is a ring in preferences $\{f\ w_1, f\ w_2, g\ w_1w_2\}$ that satisfies the following: In each ring coalition, there is one agent who prefers a stable coalition to the ring coalition and another agent who prefers the ring coalition to a *stable coalition (possibly the same one).*[13] *Therefore, by Proposition 3, this stable game induced by Kalai-Smorodinsky bargaining lacks convergence to stability. The non-trivial absorbing set is formed by the coalition structures* $\{f\,w_1, g\,w_2\}$, $\{g, f\,w_2, w_1\}$, $\{f\,w_2, g\,w_1\}$, $\{f, g\,w_1w_2\}$, *and* $\{g, f\,w_1, w_2\}$. $\Diamond$

5.2 Coalition formation games and rationing rules

In the model considered by Gallo and Inarra (2018), there is a set of agents with claims and each coalition of agents produces an output which is insufficient to meet the claims of its members. Formally, given set of agents N and a set of coalitions $\mathcal{K} \subseteq 2^N \setminus \{\emptyset\}$, a *coalitional rationing problem* is a tuple $(d_N, y(C)_{C\in\mathcal{K}})$ where $d_N = (d_i)_{i\in N} \in \mathbb{R}^N_+$ is a claims vector, $y(C) \in \mathbb{R}_+$ is the output of coalition C and $\sum_{i\in C} d_i \geq y(C)$ for each $C \in \mathcal{K}$. Given $C \in \mathcal{K}$, the *rationing problem for* C is $(d_C, y(C))$ where $d_C = (d_i)_{i\in C}$ is the claims' vector of agents in C and $y(C)$ is the output of coalition C. An *allocation* for the rationing problem $(d_C, y(C))$ is a vector $x = (x_i)_{i\in C} \in \mathbb{R}^C_+$ such that $\sum_{i\in C} x_i = y(C)$. A *rationing rule* is a mapping that associates an allocation with each rationing problem.

Given a coalitional rationing problem $(d_N, y(C)_{C\in\mathcal{K}})$, a rationing rule F induces a coalition formation game $(N, \mathcal{K}, \succ_N)$ in the following way: for each $i \in N$ and each pair $C, C' \in \mathcal{K}$ with $i \in C \cap C'$, if $F_i(d_C, y(C)) > F_i(d_{C'}, y(C'))$ then $C \succ_i C'$. Note that for the game to be well-defined, no pair of rationing problems should allocate the same amount to agent i.

One of the most important classes of rules for rationing problems is the class of parametric rules (see Young, 1987; Stovall, 2014). The proportional, constrained equal awards, constrained equal losses, and the Talmud and reverse Talmud rules are symmetric parametric rules while the sequential priority rule is an asymmetric parametric rule.

Let f be a collection of functions $\{f_i\}_{i\in N}$,[14] where each $f_i : \mathbb{R}_+ \times [a,b] \longrightarrow \mathbb{R}_+$ is continuous and weakly increasing in λ, $\lambda \in [a,b]$, $-\infty \leq a < b \leq \infty$

[13] This type of game is analyzed in Section 6.

[14] When the rule is symmetric, f_i is the same for all agents.

and for each $i \in N$ and $d_i \in \mathbb{R}_+$, $f_i(d_i, a) = 0$ and $f_i(d_i, b) = d_i$. Given f, a *parametric (rationing) rule* F is defined as follows. For each problem (d, y) and each $i \in N$, $F_i(d, y) = f_i(d_i, \lambda)$ where λ is chosen so that $\sum_{i \in N} f_i(d_i, \lambda) = y$.[15]

Theorem 4 *Any coalition formation game induced by a parametric rule exhibits convergence to stability.*

Proof. Let $(N, \mathcal{K}, \succ_N)$ be a coalition formation game induced by a parametric rule. Gallo and Inarra (2018) guarantee that $(N, \mathcal{K}, \succ_N)$ is a stable coalition formation game with no rings.[16] By Corollary 1, $(N, \mathcal{K}, \succ_N)$ exhibits convergence to stability. □

Gallo and Inarra (2018) characterize the class of rules that have stable coalition structures (see their Theorem 2). The random arrival rule (O'Neill, 1982) fails to guarantee stability. Moreover, focusing only on stable coalition formation games induced by the random arrival rule, we find that they may lack convergence to stability. The following example illustrates the different behavior of the proportional rule and the random arrival rule when inducing coalition formation games.[17]

[15]In the literature, f is said to be a *parametric representation* of F.

[16]Gallo and Inarra (2018) show that each parametric rationing rule induces a stable coalition formation game with no rings in preferences (Proposition 1 in Gallo and Inarra (2018)). The non-existence of rings in preferences in their setting implies the non-existence of rings in our setting.

[17] For each $C \in \mathcal{K}$, each $(d_C, y(C))$, and each $i \in C$,

Proportional rule, *Prop*:

$$Prop_i(d_C, y(C)) = \frac{d_i}{\sum_{j \in C} d_j} y(C).$$

Random arrival rule, *RA*:

$$RA_i(d_C, y(C)) = \frac{1}{|C|!} \left(\sum_{\lessdot \in \mathcal{O}^C} \min \left\{ d_i, \max \left\{ y(C) - \sum_{j \in C,\, j \lessdot i} d_j, 0 \right\} \right\} \right),$$

where $\mathcal{O}^C$ denote the class of strict orders on C, with generic element $\lessdot$.

Example 7 *Assume that there is a call to finance research projects and that a number of researchers are ready to submit a project. Each researcher has an aspiration, which depends on her CV, as to how much the money she deserves. Let* $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *be the set of researchers with the following aspirations:*

$$c_1 = c_2 = c_5 = c_7 = c_8 = c_9 = 50,\ c_3 = c_4 = c_6 = 10.$$

Researches can form various teams but participate in only one. Funding depends on the quality of the project, which in turn depends on team composition, and there is not enough money to meet the aspirations of all possible teams. Assume that the money to be assigned to each potential team is distributed according to the random arrival rule and to the proportional rule. The table below gives the coalitions, the outputs, and the distribution of the outputs given by these two rules.

Coalitions	$\{15\}$	$\{45\}$	$\{123\}$	$\{34\}$	$\{68\}$	$\{78\}$	$\{679\}$	$\{26\}$
Outputs	34	20	53	9	9	34	53	20
RA	$(17,17)$	$(5,15)$	$(\frac{73}{3},\frac{73}{3},\frac{13}{3})$	$(\frac{9}{2},\frac{9}{2})$	$(\frac{9}{2},\frac{9}{2})$	$(17,17)$	$(\frac{13}{3},\frac{73}{3},\frac{73}{3})$	$(15,5)$
Prop	$(17,17)$	$(\frac{10}{3},\frac{50}{3})$	$(\frac{265}{11},\frac{265}{11},\frac{53}{11})$	$(\frac{9}{2},\frac{9}{2})$	$(\frac{3}{2},\frac{15}{2})$	$(17,17)$	$(\frac{53}{11},\frac{265}{11},\frac{265}{11})$	$(\frac{50}{3},\frac{10}{3})$

The coalition formation game induced by random arrival rationing is:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
123	123	**34**	45	**15**	**26**	679	**78**	679
15	**26**	123	**34**	45	68	**78**	68	**9**
1	2	3	4	5	679	7	8	
					6			

In this game the stable coalition structure is $\{15, 26, 34, 78, 9\}$ *and the ring is* $\{679, 68, 78\}$*. Note that this ring together with coalitions* 123 *and* 45 *generate lack of convergence to stability through the nontrivial absorbing set formed by* $\{123, 45, 679, 8\}$, $\{123, 45, 68, 7, 9\}$, *and* $\{123, 45, 6, 78, 9\}$.

The coalition formation game induced by proportional rationing is:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
123	**123**	**123**	34	15	26	**679**	78	**679**
15	26	34	**45**	**45**	**679**	78	68	9
1	2	3	4	5	68	7	**8**	
					6			

In this game the stable coalition structure is $\{123, 45, 679, 8\}$ *and there are no rings. Therefore, the game exhibits convergence to the stable coalition structure.* ◊

6 Enclosed coalition formation games

These games exemplify real life situations in which lack of convergence to stability arises. Consider, for instance, that after an elections no single party has attained the majority required to form a government. Each party, even those with similar ideologies, often has different views about significant problems such as the degree of centrality of the country, healthcare, immigration, etc. Suppose that "the left" is fragmented into three parties and that any two of them can form a government. However, it may happen that each party refuses to govern with one of the others because of their antagonistic views about how to handle a particular problem. Although the entire left is a stable coalition, in the sense that once formed it could not be blocked by two of its parties, the situation may well end up with another election. In these games, "enclosing" a stable coalition in a ring suffices to prevent convergence to that stable coalition structure.

Definition 10 *Let* $(N, \mathcal{K}, \succ_N)$ *be a coalition formation game with a unique stable coalition structure* π^N*. A ring* $\mathcal{R}$ *is* ***enclosing*** *if the following conditions hold:*

(i) For each $R \in \mathcal{R}$ *there is a pair of agents* $i, j \in R$ *who satisfy* $\pi^N(i) \succ_i R$ *and* $R \succ_j \pi^N(j)$.

(ii) For each $R \in \mathcal{R}$ *and each* $X \in \mathcal{K} \setminus \pi^N$*, if* $X \succ R$ *then* $X \in \mathcal{R}$.

*$(N, \mathcal{K}, \succ_N)$ is said to be an **enclosed coalition formation game** when it has an enclosing ring.*

Condition (i) requires there to be one agent in each ring coalition who prefers a stable coalition to the ring coalition and another agent who prefers the ring coalition to a stable coalition (possibly the same one). This implies that there is no ring coalition that belongs to the stable partition. Condition (ii) requires that each ring coalition can be blocked only by a ring coalition.

Proposition 3 *An enclosed coalition formation game lacks convergence to stability.*

Proof. Let $(N, \mathcal{K}, \succ_N)$ be an enclosed coalition formation game with a unique stable coalition structure π^N and let $\mathcal{R}$ be its enclosing ring. Let $R \in \mathcal{R}$, and define π as follows:

$$\pi(i) = \begin{cases} R & \text{for } i \in R \\ \{i\} & \text{otherwise} \end{cases}$$

Condition (i) of Definition 10 and the fact that there is only one stable coalition structure mean that π is not stable. Call *successor of* π to each coalition structure π'' such that $\pi'' \gg^T \pi$. First, we claim that no successor of π is stable. Since π is not stable, there are π' and $C \in \mathcal{K}$ such that $\pi' \gg \pi$ via C. If $C \cap R \neq \emptyset$ then, by Condition (ii) of Definition 10, $C \in \mathcal{R}$. If $C \cap R = \emptyset$, then $R \in \pi'$. In either case, $\pi' \cap \mathcal{R} \neq \emptyset$. Therefore, by Condition (i) of Definition 10 and the uniqueness of the stable coalition structure, π' is not stable. The claim is proved is the same reasoning is applied inductively. Now we complete the proof of the proposition. Since π is not stable, by Remark 2 (iii), there are an absorbing set $\mathcal{A}^N$ and a coalition structure $\widetilde{\pi} \in \mathcal{A}^N$ such that $\widetilde{\pi} \gg^T \pi$. As $\widetilde{\pi}$ is a successor of π, $\widetilde{\pi}$ is not stable, and therefore $|\mathcal{A}^N| \geq 3$. Therefore, by Proposition 1, this enclosed coalition formation game lacks convergence to stability. □

Admittedly, this is a restricted class of coalition formation games, but it makes it clear that a small group of agents conforming an enclosing ring may be enough to preclude convergence to stability. Furthermore, important models such as matching models with complementarities and peer effects can induce enclosed coalition formation games. For instance, in the academic labor market universities frequently wish to hire academics with complementary skills so as to reinforce a specific field, and otherwise prefer to hold the offer off. On the other side of the market, for academics choosing whom to work for is an important consideration. The following example illustrates this situation.

Example 8 *Consider three universities and three professors in the academic market for economics. Candidate c_A specializes in applied economics, candidate c_B is a behaviorist and candidate c_T is a theorist. Suppose that each academic prefers a different colleague and they would rather be all together than with the least preferred colleague. Universities U_1 and U_2 can hire two candidates while university U_3 can hire all of them. Furthermore, university U_1 will hire as long as one of them is a theorist while university U_2 is not interested in this profile. Otherwise, all agents remain single. This description is consistent with the following coalition formation game:*

$\boldsymbol{U_1}$	$\boldsymbol{U_2}$	$\boldsymbol{U_3}$	$\boldsymbol{c_A}$	$\boldsymbol{c_B}$	$\boldsymbol{c_T}$
$U_1c_Ac_T$	$U_2c_Ac_B$	$\boldsymbol{U_3c_Ac_Bc_T}$	$U_1c_Ac_T$	$U_2c_Ac_B$	$U_1c_Bc_T$
$U_1c_Bc_T$	$\boldsymbol{U_2}$	U_3	$\boldsymbol{U_3c_Ac_Bc_T}$	$\boldsymbol{U_3c_Ac_Bc_T}$	$\boldsymbol{U_3c_Ac_Bc_T}$
$\boldsymbol{U_1}$			$U_2c_Ac_B$	$U_1c_Bc_T$	$U_1c_Ac_T$
			c_A	c_B	c_T

The enclosing ring $(U_1c_Ac_T, U_2c_Ac_B, U_1c_Bc_T)$ prevents convergence to the stable coalition structure $\{U_1, U_2, U_3c_Ac_Bc_T\}$. Each coalition structure of the non-trivial absorbing set contains a different ring coalition and single agents. $\diamondsuit$

Finally, the class of enclosed coalition formation games does not include but intersects with the class of weak top coalition games[18] (Banerjee et al., 2001)

[18]A coalition $W \subseteq G \subseteq N$, is a *weak top coalition* of G if it has an ordered coalition structure $(S_1, ..., S_l)$ such that (i) any agent in S_1 weakly prefers W to any subset of G and (ii) for any $k > 1$, any agent in S_k needs cooperation of at least one agent in $\cup_{m<k} S_m$ in order to form a strictly better coalition than W. A game satisfies the *weak top coalition property* if for any coalition $G \subseteq N$, there exists a weak top coalition W of G.

and the class of ordinally balanced games[19] (Bogomolnaia and Jackson, 2002). Although these classes of stable coalition formation games impose some degree of commonality on agents' preferences, guaranteeing the non-emptiness of the core, they may lack convergence to stability.

Example 9 *(see Bogomolnaia and Jackson, 2002, Section 4). Consider the following two coalition formation games:*

1	**2**	**3**
12	23	13
123	**123**	**123**
13	12	23
1	2	3

1	**2**	**3**
123	23	13
12	12	**123**
13	**123**	23
1	2	3

The game in the first table is ordinally balanced and the one in the second table satisfies the weak top coalition property. In both cases coalition $\{123\}$ *is the unique stable coalition structure and there is an enclosing ring:* $(13, 12, 23)$*. It is not possible to reach the stable coalition structure starting from any coalition structure which contains a two-agent coalition and a singleton. These two games are enclosed coalition formation games and hence do not converge to stability. If either games is modified by setting coalition* 123 *as the top choice of each agent, then the resulting game is ordinally balanced and satisfies the weak top coalition property. However, this game is not enclosed and exhibits convergence to stability.* ◊

7 Concluding remarks

To conclude, we first discuss our results considering the general class (stable and unstable) of coalition formation games and then mention some further research.

[19] A family of coalitions $\mathcal{B} \subset N$ is *balanced* if there is a vector of positive weights λ_S, such that for each agent $i \in N$, $\sum_{S \in \mathcal{B}: i \in S} \lambda_S = 1$ (see Bondareva, 1963; Shapley, 1967). A coalition formation game is *ordinally balanced* if for each balanced collection of coalitions $\mathcal{B}$ there is a coalition structure π such that for each i there is $S \in \mathcal{B}$ with $i \in S$ such that $\pi(i) \succ_i S$.

We claim that our characterization result (Theorem 2) goes beyond the analysis of convergence to stability. Our approach is certainly focused on determining what structures of preferences over coalitions generate a non-trivial absorbing set. We conclude that it is the presence of at least one effective ring component that precludes convergence to stability. Thus, our analysis is independent of whether there is a stable coalition structure (trivial absorbing set): if such a structure exists then the presence of an effective ring component precludes convergence to stability. Otherwise, the problem of convergence to stability is vacuous. However, the characterization provided identifies the agents that generate effective ring components. These agents show their dissatisfaction by blocking the ring coalitions of the effective ring component one after the other. Hence, if convergence to stability is the goal pursued then some of the coalitions formed by the dissatisfied agents must be neutralized by transforming them into non-permissible coalitions.

The notion of absorbing set can also be thought as a solution concept for coalition formation games as they always exist and show the dynamic property of outer stability. For stable coalition formation games, the coexistence of trivial and non-trivial absorbing sets is equivalent to lack of convergence to stability. For unstable coalition formation games, the dissatisfied agents of each coalition structure in a non-trivial absorbing set make any coalition structure unpredictable, but these coalition structures dominate those not in the set, so the latter are discarded as plausible outcomes.

Finally, Ballester (2004) studies the complexity of coalition formation games and shows that the computation of stable coalition structures is NP-complete. It is important to note that the size of a coalition formation game, understood as the size of the input of the program, is the size of the set of the coalition structures formed by the permissible coalitions, $|\Pi|$. NP-completeness implies that the time needed to solve a coalition formation game is likely to be exponential in $|\Pi|$ (and, of course in $|N|$). Moreover, a look at the proofs for coalition formation games in Ballester (2004) reveals that the complexity of finding the core is simply the complexity of finding the coalition structures of N inside a set of permissible coalitions. So, even if the set of coalitions is restricted, the number of possible coalition structures is still exponential and hardness is unlikely to be overcome. Furthermore, the computation for finding a ring component seems to be NP-complete since its definition depends on whether there is a non-trivial absorbing set.

References

BALLESTER, C. (2004): "NP-completeness in hedonic games," *Games and Economic Behavior*, 49, 1–30.

BANERJEE, S., H. KONISHI, AND T. SÖNMEZ (2001): "Core in a simple coalition formation game," *Social Choice and Welfare*, 18, 135–153.

BOGOMOLNAIA, A. AND M. O. JACKSON (2002): "The stability of hedonic coalition structures," *Games and Economic Behavior*, 38, 201–230.

BONDAREVA, O. N. (1963): "Some applications of linear programming methods to the theory of cooperative games," *Problemy Kibernetiki*, 10, 119–139.

CHUNG, K.-S. (2000): "On the existence of stable roommate matchings," *Games and Economic Behavior*, 33, 206–230.

DEMUYNCK, T., P. J.-J. HERINGS, R. D. SAULLE, AND C. SEEL (2019): "The myopic stable set for social environments," *Econometrica*, 87, 111–138.

DIAMANTOUDI, E., E. MIYAGAWA, AND L. XUE (2004): "Random paths to stability in the roommate problem," *Games and Economic Behavior*, 48, 18–28.

DIAMANTOUDI, E. AND L. XUE (2003): "Farsighted stability in hedonic games," *Social Choice and Welfare*, 21, 39–61.

DREZÉ, J. H. AND J. GREENBERG (1980): "Hedonic coalitions: optimality and stability," *Econometrica*, 48, 987.

Echenique, F. and M. B. Yenmez (2007): "A solution to matching with preferences over colleagues," *Games and Economic Behavior*, 59, 46–71.

Eriksson, K. and O. Häggström (2008): "Instability of matchings in decentralized markets with various preference structures," *International Journal of Game Theory*, 36, 409–420.

Gale, D. and L. S. Shapley (1962): "College admissions and the stability of marriage," *The American Mathematical Monthly*, 69, 9–15.

Gallo, O. and E. Inarra (2018): "Rationing rules and stable coalition structures," *Theoretical Economics*, 13, 933–950.

Hart, S. and M. Kurz (1983): "Endogenous formation of coalitions," *Econometrica*, 1047–1064.

Iehlé, V. (2007): "The core-partition of a hedonic game," *Mathematical Social Sciences*, 54, 176–185.

Inal, H. (2015): "Core of coalition formation games and fixed-point methods," *Social Choice and Welfare*, 45, 745–763.

Inarra, E., C. Larrea, and E. Molis (2013): "Absorbing sets in roommate problems," *Games and Economic Behavior*, 81, 165–178.

Jackson, M. O. and A. Watts (2002): "The evolution of social and economic networks," *Journal of Economic Theory*, 106, 265–295.

Kalai, E., A. Postlewaite, J. Roberts, et al. (1979): "A group incentive compatible mechanism yielding core allocations," *Journal of Economic Theory*, 20, 13–22.

Kalai, E. and D. Schmeidler (1977): "An admissible set occurring in various bargaining situations," *Journal of Economic Theory*, 14, 402–411.

Kalai, E. and M. Smorodinsky (1975): "Other solutions to Nash's bargaining problem," *Econometrica*, 43, 513–518.

KLAUS, B. AND F. KLIJN (2005): "Stable matchings and preferences of couples," *Journal of Economic Theory*, 121, 75–106.

KNUTH, D. (1976): "Marriages stable. université de montréal press, translated as "stable marriage and its relation to other combinatorial problems,"," *CRM Proceedings and Lecture Notes, American Mathematical Society*.

KOJIMA, F. AND M. U. ÜNVER (2008): "Random paths to pairwise stability in many-to-many matching problems: a study on market equilibration," *International Journal of Game Theory*, 36, 473–488.

MAULEON, A., N. ROEHL, AND V. VANNETELBOSCH (2019): "Paths to stability for overlapping group structures," *Journal of Mathematical Economics*, 83, 19–24.

MYERSON, R. B. (1977): "Graphs and cooperation in games," *Mathematics of operations research*, 2, 225–229.

NASH, J. (1950): "The bargaining problem," *Econometrica*, 28, 155–162.

OLAIZOLA, N. AND F. VALENCIANO (2014): "Asymmetric flow networks," *European Journal of Operational Research*, 237, 566–579.

O'NEILL, B. (1982): "A problem of rights arbitration from the Talmud," *Mathematical Social Sciences*, 2, 345–371.

PÁPAI, S. (2003): "Random paths to stability in Hedonic coalition formation," *University of Notre Dame working paper*.

PYCIA, M. (2012): "Stability and preference alignment in matching and coalition formation," *Econometrica*, 80, 323–362.

RAY, D. AND R. VOHRA (2015a): "Coalition formation," in *Handbook of game theory with economic applications*, Elsevier, vol. 4, 239–326.

——— (2015b): "The farsighted stable set," *Econometrica*, 83, 977–1011.

ROTH, A. E. AND M. SOTOMAYOR (1990): "Two sided matching: a study in game-theoretic modeling and analysis," *Econometric Society Monographs*, 18.

ROTH, A. E. AND J. H. VANDE VATE (1990): "Random paths to stability in two-sided matching," *Econometrica*, 1475–1480.

SCHWARTZ, T. (1970): "On the possibility of rational policy evaluation," *Theory and Decision*, 1, 89–106.

SHAPLEY, L. S. (1967): "On balanced sets and cores," *Naval Research Logistics Quarterly*, 14, 453–460.

SHENOY, P. P. (1979): "On coalition formation: a game-theoretical approach," *International Journal of Game Theory*, 8, 133–164.

——— (1980): "A dynamic solution concept for abstract games," *Journal of Optimization Theory and Applications*, 32, 151–169.

STOVALL, J. E. (2014): "Collective rationality and monotone path division rules," *Journal of Economic Theory*, 154, 1–24.

TAMURA, A. (1993): "Transformation from arbitrary matchings to stable matchings," *Journal of Combinatorial Theory, Series A*, 62, 310–323.

TAN, J. J. (1991): "A necessary and sufficient condition for the existence of a complete stable matching," *Journal of Algorithms*, 12, 154–178.

YOUNG, H. P. (1987): "On dividing an amount according to individual claims or liabilities," *Mathematics of Operations Research*, 12, 398–414.